Hamilton–Jacobi's equation and Arnold's diffusion near invariant tori in a priori unstable isochronous systems.

G. Gallavotti
Fisica, Roma1

Abstract: *Local integrability of hyperbolic oscillators is discussed to provide an introductory example of the Arnold's diffusion phenomenon in a forced pendulum.*

Keywords: Hamilton Jacobi, KAM, Arnold diffusion, invariant tori

1. Introduction.

Here I expose, in a simple case, a well established technique. I shall show that a Hamiltonian system composed of two clocks (*i.e.* isochronous rotators) and a hyperbolic oscillator can be integrated by quadratures in a full neighborhood of the unperturbed equilibrium of the oscillator. The rotators angular velocities form a vector $\underline{\omega} = (\omega_1, \omega_2)$ which we suppose Diophantine, see §2.

The system is described by $\ell - 1$ pairs $(\underline{A}, \underline{\alpha}) \in R^{\ell-1} \times T^{\ell-1}$ of "actions" $\underline{A} = (A_1, \ldots, A_\ell)$ and "angles" $(\alpha_1, \ldots, \alpha_{\ell-1})$ and *one* pair of conjugate coordinates (p, q) with a Hamiltonian:

$$\underline{\omega} \cdot \underline{A} + J(p\,q) + \varepsilon f(\underline{\alpha}, p, q) \tag{1.1}$$

with $J(x)$ and $f(\underline{\alpha}, p, q)$ analytic for $|x| < \kappa$, $|p|, |q| < \kappa^{1/2}$ and $\underline{\alpha} \in T^{\ell-1}$. We suppose that $J'(0) > 0$ and that $\kappa > 0$ is small enough so that for some $\overline{g}_0 > 0$ it is $\overline{g}_0 < |\mathrm{Re}\, J'(x)| \le |J'(x)| < 2\overline{g}_0$ for all $|x| < \kappa$.

The Hamilton Jacobi's equation for the integration of the system is:

$$\underline{\omega} \cdot \partial_{\underline{\alpha}} \Phi + J((p' + \partial_q \Phi) q) + \varepsilon f(\underline{\alpha}, p' + \partial_q \Phi, q) = \tilde{J}(p'(q + \partial_{p'} \Phi)) \tag{1.2}$$

where the unknowns are $\Phi(\underline{\alpha}, p', q)$, $\tilde{J}(x)$, and we require that $\Phi, \tilde{J} - J$ be divisible by ε.

If $\Phi, \tilde{J}$ exist and are analytic in their arguments and in ε, it follows that for ε small enough there is a canonical change of coordinates transforming the Hamiltonian (1.1) into:

$$\underline{\omega} \cdot \underline{A}' + \tilde{J}(p'q') \tag{1.3}$$

hence the points with $p' = q' = 0$ form a family of invariant tori parameterized by $\underline{A}'$. And the motions are completely known, and very simple, in the new coordinates.

Theorem 1: *the Hamilton Jacobi's equation (1.1) admits an analytic solution for all ε small enough. The solution is analytic in ε and in p', q, divisible by ε and so is $\tilde{J} - J$.*

This (form of a well known) theorem is very useful because it provides *full control* of motions dwelling a very long time around the invariant tori, in spite of the nonlinearity of the equations. This turns out to be the key feature needed in the theory of Arnold's diffusion.

In §2 I prove the theorem (following [CG] where, however, it is discussed in a much more general case). In §3 I introduce the notion of Arnold's diffusion, [A], in the example of a quasi–periodically forced pendulum (*i.e.* in a system consisting of a pendulum in interaction with two clocks) and discuss how to apply the theorem to show its existence. In §4,§5 I provide the details, from [G3].

The example may be useful as an introduction and to clarify the intrinsic simplicity of a problem that was solved by Arnold many years ago and that, outside a small circle of specialists, still seems to be (unreasonably) presented as a very hard problem.

§2. A classical perturbative analysis.

Consider the Hamiltonian:

$$\mathcal{H}_0 = \underline{\omega} \cdot \underline{A}_0 + J_0(x_0) + f_0(\underline{\alpha}, p_0, q_0) \tag{2.1}$$

with J_0 holomorphic in $Q_{\kappa_0} = \{|x_0| < \kappa_0\}$ and f_0 holomorphic in $P_{\xi_0,\kappa_0} = \{e^{-\xi_0} < |e^{i\alpha_j}| < e^{\xi_0}, \text{for } j = 1, \ldots \ell-1\} \times \{|p_0|, |q_0| < \kappa_0^{1/2}\}$. Here ℓ is the total number of degrees of freedom. We assume that $\underline{\omega}$ enjoys a *Diophantine property*: $|\underline{\omega} \cdot \underline{\nu}| > C_0|\underline{\nu}|^{-\tau}$ for some $C_0, \tau > 0$ and for all non zero integer component vectors $\underline{\nu} = (\nu_1, \ldots, \nu_{\ell-1}) \in Z^{\ell-1}$.

We call $\varepsilon_0 \stackrel{def}{=} ||f_0||_{\xi_0,\kappa_0}$ the *lowest upper bound* of $|f_0|$ in its domain of definition and we call $\tilde{g}_0 = ||J_0'||_{\kappa_0}$ the lowest upper bound to the derivative of J_0 in its domain; we also need the greatest lower bound to $\mathrm{Re}\, J_0$ that we call $\overline{g}_0$. We set $\Gamma_0 = \min\{\overline{g}_0, C_0\}$ and by the assumptions following (1.1) it is $\overline{g}_0 < |\mathrm{Re}\, J'(x)| \le |J'(x)| < 2\overline{g}_0$. Summarizing the constants $\varepsilon_0, \overline{g}_0, \Gamma_0$ are defined so that:

$$\varepsilon_0 = ||f_0||_{\xi_0,\kappa_0}, \qquad \overline{g}_0 < |\mathrm{Re}\, J'(x)| \le |J'(x)| < 2\overline{g}_0, \qquad \Gamma_0 = \min\{\overline{g}_0, C_0\} \tag{2.2}$$

The system has, therefore, two time scales, namely the pendulum time scale $\overline{g}_0^{-1}$ and the clocks time scale C_0^{-1}: Γ_0 is the smallest.

Let us define the "generating function" $\underline{A}_1 \cdot \underline{\alpha} + p_1 q_0 + \Phi_0(\underline{\alpha}, p_1, q_0)$ via a function $\Phi_0(\underline{\alpha}, p, q) = -(g_0(x)\,\partial\!\!\!/ + \underline{\omega} \cdot \underline{\partial})^{-1} \tilde{f}_0(\underline{\alpha}, p, q)$ where:

- we set $g_0(x)$ equal to the derivative $J_0'(x)$ of $J_0(x)$: $g_0(x) \stackrel{def}{=} J_0'(x)$.
- we set $\partial\!\!\!/ \stackrel{def}{=} q\partial_q - p\partial_p$ and $\underline{\partial} \stackrel{def}{=} \partial_{\underline{\alpha}}$.
- $\tilde{f}_0(\underline{\alpha}, p, q) = f_0(\underline{\alpha}, p, q) - \overline{f}_0(pq)$ and $\overline{f}_0(pq)$ is obtained from $f(\underline{\alpha}, p, q)$ by taking the average over $\underline{\alpha}$ and then expanding in powers of p, q the result and keeping only the terms that depend on the product pq. This means that we write $f(\underline{\alpha}, p, q) = \sum_{m,n=0}^{\infty} f_{m,n}(\underline{\alpha}) p^m q^n = \sum_{m,n,\underline{\nu}} f_{\underline{\nu},m,n} e^{i\underline{\nu}\cdot\underline{\alpha}} p^m q^n$ and set $\overline{f}(x) = \sum_m f_{\underline{0},m,m} x^m$. In this definition we keep control of the size of $\overline{f}$ because $|f_{m,n}(\underline{\alpha})| < \varepsilon_0 \kappa_0^{-\frac{1}{2}(m+n)}$ and $|f_{\underline{\nu},m,n}| < \varepsilon_0 \kappa_0^{-\frac{1}{2}(m+n)} e^{-\xi_0|\underline{\nu}|}$, where $|\underline{\nu}| = \sum_j |\nu_j|$.

Under the conditions on the auxiliary parameter $\delta_0 > 0$ specified below and in the smaller domain $P_{\xi_0-\delta_0,\kappa_0 e^{-\delta_0}}$, the generating map:

$$p_0 = p_1 + \partial_{q_0}\Phi_0(\underline{\alpha}, p_1, q_0),\;\; q_1 = q_0 + \partial_{p_1}\Phi_0(\underline{\alpha}, p_1, q_0),\;\; \underline{A}_0 = \underline{A}_1 + \partial_{\underline{\alpha}}\Phi_0(\underline{\alpha}, p_1, q_0) \tag{2.3}$$

defines a canonical map $\mathcal{C}_0 : P_{\xi_0-\delta_0,\kappa_0 e^{-\delta_0}} \to P_{\xi_0,\kappa_0}$:

$$\mathcal{C}_0(\underline{A}_1, p_1, q_1) = \begin{cases} \underline{A}_0 = \underline{A}_1 + \underline{H}_1(\underline{\alpha}, p_1, q_1) \\ p_0 = p_1 + L_1(\underline{\alpha}, p_1, q_1) \\ q_0 = q_1 + \tilde{L}_1(\underline{\alpha}, p_1, q_1) \end{cases} \tag{2.4}$$

where $\underline{H}_1, L_1, \tilde{L}_1$ are suitable functions holomorphic in $P_{\xi_0-\delta_0,\kappa_0 e^{-\delta_0}}$.

In fact by simple dimensional estimates ("Cauchy estimates"), setting $\xi_0' = \xi_0 - \frac{1}{2}\delta_0$, and $\kappa_0' = \kappa_0 e^{-\frac{1}{2}\delta_0}$ and choosing B_0 large enough it is (recall the above definition $\varepsilon_0 \stackrel{def}{=} ||f_0||_{\xi_0,\kappa_0}$; see also appendix A2):

$$\begin{aligned} &||\partial_q\Phi_0||_{\xi_0',\kappa_0'}\,,\; ||\partial_p\Phi_0||_{\xi_0',\kappa_0'},\; \kappa_0^{-\frac{1}{2}}||\partial_{\underline{\alpha}}\Phi_0||_{\xi_0',\kappa_0'} \le B_0 {\Gamma_0}^{-1} \kappa_0^{-\frac{1}{2}} \delta_0^{-\ell-\tau-1}\varepsilon_0 \\ &||\partial_p\partial_q\Phi_0||_{\xi_0',\kappa_0'} \le B_0 {\Gamma_0}^{-1} {\kappa_0}^{-1} \delta_0^{-\ell-\tau-2}\varepsilon_0 \end{aligned} \tag{2.5}$$

hence under the condition $B_1\Gamma_0^{-1}\kappa_0^{-1}\varepsilon_0\delta_0^{-\ell-\tau-2}<1$, with B_1 large enough, the map $\mathcal{C}_0$ is defined (by the implicit functions theorem) and maps $P_{\xi_0-\delta_0,\kappa_0e^{-\delta_0}}$ into P_{ξ_0,κ_0}.

The choice of Φ_0 is just right so that by evaluating the Hamiltonian in the new coordinates it takes the form:

$$\mathcal{H}_1=\underline{\omega}\cdot\underline{A}_1+J_1(p_1q_1)+f_1(\underline{\alpha},p_1,q_1) \tag{2.6}$$

with $J_1(x)=J_0(x)+\overline{f}_0(x)$ and a suitable f_1 "*which is of order* $||f_0||^2$".

In fact set $\xi_1=\xi_0-2\delta_0$, $\kappa_1=\kappa_0e^{-2\delta_0}$ and $\overline{g}_1=\overline{g}_0-B_2\varepsilon_0\kappa_0^{-1}\delta_0^{-2}$, $\tilde{g}_1=2\overline{g}_0+B_2\varepsilon_0\kappa_0^{-1}\delta_0^{-2}$, with B_2 large (see below).

Note that $B_2\varepsilon_0\kappa_0^{-1}\delta_0^{-2}$ is a dimensional bound for $\frac{d}{dx}\overline{f}(x)$ for $|x|<\kappa_0e^{-\delta_0/2}$ (the δ_0^{-2} is due to the fact that the series over p,q is a double series: hence isolating the terms $(pq)^m$ and trying to bound the series expressing $\overline{f}$ (via the bounds on the Fourier–Taylor coefficients $f_{\underline{\nu},m,n}$ gives a factor $(1-e^{-\delta_0/2})^{-1}$, and furthermore one has to differentiate $\overline{f}$ with respect to x which yields a further division by $\kappa_0(1-e^{-\delta_0/2})$. Then:

$$\overline{g}_1<|\mathrm{Re}\,J_1'|\le|J_1'|<\tilde{g}_1,\qquad \varepsilon_1=||f_1||_{\xi_1,\kappa_1}<B_3\frac{\varepsilon_0^2}{\Gamma_0\kappa_0}\frac{\overline{g}_0}{\Gamma_0}\delta_0^{-2(\ell+\tau)-4} \tag{2.7}$$

provided $B_4\varepsilon_0\Gamma_0^{-1}\kappa_0^{-1}\delta_0^{-\ell-\tau-1}<1$ for some B_4 large enough and $\overline{g}_1>\frac{1}{2}\overline{g}_0,\tilde{g}_1<4\overline{g}_0$. The bound (2.7) can be checked by a first order expansion of the Hamiltonian $\mathcal{H}_0$ evaluated by expressing p_0, $\underline{A}_1$ in terms of $p_1,q_0,\underline{\alpha}$ via (2.3) and by developing to first order with respect to the increments p_0-p_1, q_1-q_0, $\underline{A}_0-\underline{A}_1$. The choice of Φ_0 is what is needed to eliminate the first order terms and the remainders are easily estimated via Cauchy estimates, in a domain slightly smaller than the one in which $\mathcal{C}_0$ is defined, which we recall to be $P_{\xi_0-\delta_0,\kappa_0e^{-\delta_0}}$. For details see Appendix A2 below (see also [G1], Ch. V, §12).

Proceeding as usual in the KAM proofs, [G1], define $\xi_j=\xi_0-2\delta_0-\ldots-2\delta_{j-1}$ and $\kappa_j=\kappa_0e^{-2\delta_0-\ldots-2\delta_{j-1}}$ and fix $\delta_j=\xi_0((j+1)4)^{-2}$. Then it is: $\kappa_j\ge\kappa_0e^{-\frac{1}{2}\xi_0}$ and $\xi_j\ge\frac{1}{2}\xi_0$. We also set $\Gamma_j=\min\{\overline{g}_j,C_0\}$ and $\tilde{g}_{j+1}=\tilde{g}_j+B_2\varepsilon_j\kappa_j^{-1}\delta_j^{-2(\ell+\tau)-4}$.

In this way *under the conditions* $B_4\varepsilon_j\Gamma_0^{-1}\kappa_0^{-1}\delta_j^{-\ell-\tau-1}<1$ *and* $2\overline{g}_0>\tilde{g}_j\ge\overline{g}_j\ge\frac{1}{2}\overline{g}_0$ it will be: $\varepsilon_{j+1}\le B_5\varepsilon_j^2\Gamma_0^{-1}\kappa_0^{-1}\overline{g}_0\Gamma_0^{-1}\xi_0^{-q}\,(j+1)^q$ having set $q\stackrel{def}{=}2(\ell+\tau)+4$.

Defining $\lambda_{j+1}=\lambda_j^2(j+1)^{-q}$ for $j\ge1$ and $\lambda_0=1$ we get $\lambda_j\ge e^{-q_02^j}$ with a q_0 bounded proportionally to q. Furthermore defining, for a suitably large B_5, $\eta_j=B_5\overline{g}_0\Gamma_0^{-1}\xi_0^{-q}\,\lambda_j\varepsilon_j$ we find, from the definition of λ_j:

$$\eta_j\le\eta_0^{2^j} \tag{2.8}$$

Therefore, if B_6 is suitably large so that the condition:

$$B_6\varepsilon_0\Gamma_0^{-1}\overline{g}_0\Gamma_0^{-1}\kappa_0^{-1}\xi_0^{-q}<1,\qquad q=2(\ell+\tau)+4 \tag{2.9}$$

encompasses all the ones found so far, one can define not only $\mathcal{C}_0$ but also, recursively, $\mathcal{C}_j$ in the domain P_{ξ_j,κ_j}, with image in $P_{\xi_{j-1},\kappa_{j-1}}$.

The composition $\mathcal{C}\stackrel{def}{=}\ldots\circ\mathcal{C}_2\circ\mathcal{C}_1\circ\mathcal{C}_0$ of the canonical maps will be well defined and it will give as a result a map $\mathcal{C}:\,(\underline{A}_0,p_0,q_0)\longleftrightarrow(\underline{A}_\infty,p_\infty,q_\infty)$ still of the form (2.4) with some function $\underline{H},L,\tilde{L}$ analytic in $P_{\frac{1}{2}\xi_0,\kappa_0e^{-\frac{1}{2}\xi_0}}$ and transforming the Hamiltonian (2.1) into the "normal form":

$$\mathcal{H}_\infty=\underline{\omega}\cdot\underline{A}_\infty+J_\infty(p_\infty q_\infty) \tag{2.10}$$

with J_∞ holomorphic in $Q_{\kappa_0e^{-\frac{1}{2}\xi_0}}$ wit $\frac{1}{2}\overline{g}_0<|\mathrm{Re}\,J_\infty'|\le|J_\infty'|<4\overline{g}_0$.

Comments: if $f_0(\underline{\alpha},p,q)=f(-\underline{\alpha},q,p)$ then the symmetries of the problem imply that:

$$\underline{H}_\infty(\underline{\alpha},p,q)=\underline{H}_\infty(-\underline{\alpha},q,p),\;\; L_\infty(\underline{\alpha},p,q)=\tilde{L}_\infty(-\underline{\alpha},q,p) \tag{2.11}$$

a limit as $j\to\infty$ of the corresponding relations which also hold identically for $\underline{H}_j, L_j, \tilde L_j$, for $j=1,2,\ldots$. Furthermore $\underline{H}$ has by construction $\underline{0}$ average when evaluated at $p=q=0$ (a property of "twistless tori", see [G2]) or, more generally, at $pq=0$. I shall call, therefore, $\underline{A}_\infty$ the *average actions* of such torus.

§3. An application.

We consider a system with Hamiltonian:

$$\mathcal{H}=\underline{\omega}\cdot\underline{A}+\frac{I^2}{2}+g^2(\cos\varphi-1)+\varepsilon f(\varphi,\underline{\alpha}) \tag{3.1}$$

where $(I,\varphi)\in R\times T^1$ describe a pendulum while $(\underline{A},\underline{\alpha})\in R^2\times T^2$ describe two clocks and f is an interaction which we take to be a even trigonometric polynomial in $\underline{\alpha}=(\alpha_1,\alpha_2),\varphi$. A non trivial case is $f=\cos(\alpha_1+\varphi)+\cos(\alpha_2+\varphi)$.

Near the pendulum equilibrium one can use Jacobi's coordinates p,q, see Appendix A1. In such *local coordinates* the system has Hamiltonian:

$$\mathcal{H}'=\underline{\omega}\cdot\underline{A}+J(pq)+\varepsilon f(\underline{\alpha},p,q) \tag{3.2}$$

Hence by the theorem above we can find a (local) system of coordinates $\underline{A}_\infty,\underline{\alpha},p_\infty,q_\infty$ in which the motion is described by a trivial Hamiltonian (2.10). This means that it is:

$$\underline{A}_\infty=const,\ \ \underline{\alpha}\to\underline{\alpha}+\underline{\omega}t,\ \ p_\infty\to p_\infty e^{-\tilde g(x)t},\ \ q_\infty\to q_\infty e^{\tilde g(x)t} \tag{3.3}$$

which shows that the set $p_\infty=q_\infty=0$ is an invariant torus parameterized by $\underline{A}_\infty$ and denoted $\mathcal{T}(\underline{A}_\infty)$.

Physically this is a very special set of motions in which the pendulum *does not ever fall down from its unstable equilibrium position* but performs small oscillations around it. It also shows that the sets $q=0$, denoted $W^s(\underline{A}_\infty)$, or $p=0$, denoted $W^u(\underline{A}_\infty)$, are invariant manifolds for $t>0$ and $t<0$ respectively, such that data on them evolve towards the (quasi periodic) motions on the invariant torus $\mathcal{T}(\underline{A}_\infty)$, respectively, as $t\to\pm\infty$.

The meaning of $\underline{\omega}\cdot\underline{A}$ is the "energy of the springs that move the clocks". *The problem of Arnold's diffusion is whether one can find motions whose net effect is to transfer energy from one reservoir to the other for all $\varepsilon>0$ no matter how small*: note that if $\varepsilon=0$ transfer is impossible.

Mathematically this is a motion that starts near a quasi periodic motion with average action $\underline{A}=\underline{A}_0$ and ends up, after a finite time, near one with average action $\underline{A}_1\ne\underline{A}_0$. The key point in this definition is that Arnold's diffusion between $\underline{A}_0$ and $\underline{A}_1$ if such motions exist *for all $\varepsilon>0$ small enough*, no matter how small (provided $\varepsilon\ne0$ and, of course, for suitable ε–dependent initial data). Here "near" means closer than the half the distance between the tori corresponding to $\underline{A}_0$ and $\underline{A}_1$.

Of course for this to be possible it must be that the extreme motions have the same energy: it is a simple *symmetry property* of the above models (with f even) that the condition that two quasi periodic motions with parameters $\underline{A}_\infty=\underline{A}_0$ and $\underline{A}_\infty=\underline{A}_1$ have the same energy is, generically and if ε is small, simply $\underline{\omega}\cdot(\underline{A}_1-\underline{A}_0)=0$, see [G3].

Consider the line $s\to\underline{A}(s)=\underline{A}_0+(\underline{A}_1-\underline{A}_0)s$, $s\in[0,1]$, supposing it orthogonal to $\underline{\omega}$. One says that there is a heteroclinic chain between $\underline{A}_0$ and $\underline{A}_1$ if there are $\mathcal{N}$ values $s_0=0,s_1,\ldots s_{\mathcal{N}}$ such that for all j's there exists a "heteroclinic" motion, *i.e.* a motion that is asymptotic as $t\to-\infty$ to the quasi periodic motion on the torus with average actions $\underline{A}_\infty=\underline{A}(s_j)$ and that is asymptotic as $t\to+\infty$ to the quasi periodic motion on the torus with average actions $\underline{A}_\infty=\underline{A}(s_{j+1})$.

It is a simple application of the implicit functions theorem to show that any chain with $s_{j+1}-s_j$ small enough is a heteroclinic chain, see [G3], for generic f. In fact if

$f = \cos(\alpha_1+\varphi)+\cos(\alpha_2+\varphi)$ this is the case, see [CG], [GGM]. The number $\mathcal{N}$ will be in this case, and in general, of size $O(\varepsilon^{-1})$, see [GGM].

How small ε has to be so that it is "small enough"? this depends on g. An interesting question is how small has ε to be compared to g (the pendulum characteristic frequency).

Consider a simple case: $\underline{\omega}=(\eta,1)$ and $g^2=\eta$ and $f(\underline{\alpha},\varphi)=\beta\cos(\alpha_2+\varphi)+\varepsilon\eta^c\cos(\alpha_1+\varphi)$. In this case if c is large enough taking $\beta=\varepsilon\eta^c$ is sufficient, see [GGM]. However one can do much better and allow β to be larger than 1, *i.e. independent of* ε! provided one excludes *finitely many* possibly exceptional values of β, see [GGM]. This shows that a often stated condition that β has to be exponentially small in $\eta^{-1/2}$ can be improved in three time scales problems.

We now study the existence of diffusion reproducing for convenience of the reader §4 and §5 of [G3].

§4. Geometric concepts.

Let $2\,\kappa>0$ be smaller than the radius of the disk in the (p,q) plane where the canonical coordinates $(\underline{A}_\infty,\underline{\alpha},p_\infty,q_\infty)$ can be used. I will drop soon the subscripts ∞ to simplify the notation. To visualize the geometry of the problem we shall need the following geometric objects:

•(a) a point X_i, heteroclinic between the torus parameterized by $\underline{A}_\infty=\underline{A}(s_i)\stackrel{def}{=}\underline{A}_i$ which we shall denote $\mathcal{T}(\underline{A}_i)$ and the "next" torus $\mathcal{T}(\underline{A}_{i+1})$: this is a point X_i such that $S_tX_i\xrightarrow[t\to+\infty]{}\mathcal{T}(\underline{A}_{i+1})$ and $S_tX_i\xrightarrow[t\to-\infty]{}\mathcal{T}(\underline{A}_i)$. We denote the local coordinates of X_i as $X_i=(\underline{A}_i,\underline{\alpha}_i,0,\kappa)$.

•(b) we can extend the manifolds $W^s(\mathcal{T}(\underline{A}_i))$ and $W^u(\mathcal{T}(\underline{A}_i))$ by applying to them the time evolution flow S_t. Then by our assumption the $\underline{A}_i$ form a heteroclinic chain and therefore $W^s(\mathcal{T}(\underline{A}_{i+1}))$ evolves in a finite (negative) time to the vicinity of $\mathcal{T}(\underline{A}_i)$, and in fact its image (at a properly chosen negative time) will contain X_i. Hence it can be described in the local coordinates near such torus. The equations, at fixed $q=\kappa$, of the connected part of $W^s(\underline{A}_{i+1})$ containing X_i, will be written in the local coordinates near $\mathcal{T}(\underline{A}_i)$ as:

$$Y_i(\underline{\alpha})=(\underline{A}^s_{i+1}(\underline{\alpha}),\underline{\alpha},p^s_{i+1}(\underline{\alpha}),\kappa) \tag{4.1}$$

with $|\underline{\alpha}-\underline{\alpha}_i|<\zeta$ for some $\zeta>0$ (i–independent): it is $\underline{A}^s_{i+1}(\underline{\alpha}_i)=\underline{A}_i$, $p^s_{i+1}(\underline{\alpha}_i)=0$ because we require $Y_i(\underline{\alpha}_i)=X_i$. There are constants F',F such that $|\underline{A}^s_{i+1}(\underline{\alpha})-\underline{A}^s_{i+1}(\underline{\alpha}_i)|$ and $\max_{|\underline{\alpha}-\underline{\alpha}_i|=fixed}|p^s_{i+1}(\underline{\alpha})|$ are bounded, for ζ small enough, below by $F'|\underline{\alpha}-\underline{\alpha}_i|$ and above by $F|\underline{\alpha}-\underline{\alpha}_i|$; the constants F',F are generically positive, see [GGM] for instance. They are positive in the (non trivial) example $f=\cos(\alpha_1+\varphi)+\cos(\alpha_2+\varphi)$.

Note that $W^s(\underline{A}_{i+1})$ also contains a part with local equations $(\underline{A}_{i+1},\underline{\alpha},p,0)$ which is *not* to be confused with the previous one described by the function $Y_i(\underline{\alpha})$. This is more easily understood by looking at the meaning of the above objects in the original $(\underline{A},\underline{\alpha},I,\varphi)$ coordinates: in a way the first part of $W^s(\underline{A}_{i+1})$ is close to $\varphi=0$ and the second to $\varphi=2\pi$. They can be close to each other because of the periodicity of φ, but they are conceptually quite different.

•(c) a point $P_i=Y_i(\tilde{\underline{\alpha}}_i)$ with $|\tilde{\underline{\alpha}}_i-\underline{\alpha}_i|=r_i$, where $\tilde{\underline{\alpha}}_i,r_i$ will be determined recursively, and a neighborhood B_i:

$$B_i=\{|\underline{A}-\underline{A}^s_{i+1}(\underline{\alpha})|<\rho_i,\ |\underline{\alpha}-\tilde{\underline{\alpha}}_i|<\rho_i,\ |p^s_{i+1}(\underline{\alpha})-p|<\rho_i,\ q=\kappa\} \tag{4.2}$$

where $\rho_i<r_i$ is another length to be determined recursively. If $\overline{g}\stackrel{def}{=}\frac{1}{2}\overline{g}_0,4\overline{g}_0$ are a lower and upper bound to $\tilde{J}'(x)$ for $|x|<4\kappa^2$, the point P_i evolves in a time $T_i\simeq\overline{g}^{-1}\log\kappa^{-1}$ into a point X'_i near $\mathcal{T}(\underline{A}'_{i+1})$ which has local coordinates $X'_i=(\underline{A}_{i+1},\underline{\alpha}'_i,\kappa,0)$.

•(d) The points ξ of the set B_i are mapped by the time evolution to points that, at the beginning at least, come close to $\mathcal{T}(\underline{A}_{i+1})$ and in a time $\tau(\xi)$ acquire local coordinates near $\mathcal{T}(\underline{A}_{i+1})$ with $p=\kappa$ exactly: the time $\tau(\xi)$ is of the order of $\overline{g}^{-1}\log\kappa^{-1}$.

If S_t is the time evolution flow for the system (1.1) we write $S\xi = S_{\tau(\xi)}\xi$ (note that S depends also on i). Then S maps the set B_i into a set SB_i containing:

$$B'_i = \{|\underline{A}-\underline{A}_{i+1}| < \frac{1}{E}\rho_i,\ |\underline{\alpha}-\underline{\alpha}'_i| < \frac{1}{E}\rho_i,\ p=\kappa,\ |q| < \frac{1}{E}\rho_i\} \tag{4.3}$$

because all the points in B_i with $\underline{A}=\underline{A}^s_{i+1}(\underline{\alpha})$, $p=p^s_{i+1}(\underline{\alpha})$, $q=\kappa$ evolve to points with $\underline{A}=\underline{A}_{i+1}$, $p=\kappa$, $q=0$ and $\underline{\alpha}$ close to $\underline{\alpha}'_i$, by the definitions. Here E is a bound on the jacobian matrix of S (which, being essentially a flow over a time $O(\overline{g}^{-1}\log\kappa^{-1})$, has derivatives bounded i–independently: since we suppose that ε is "small enough" we could take $E=1+b\varepsilon$ for some $b>0$ if $|\underline{A}_i-\underline{A}_{i+1}|<O(\varepsilon)$).

§5. The [CG]-method of proof of the theorem.

Let $\mathcal{N}$ be the number of elements of the heteroclinic chain. Consider the points $Y^s_{i+1}(\underline{\alpha}) \in W^s(\underline{A}_{i+2})$ with coordinates $(\underline{A}^s_{i+2}(\underline{\alpha}), \underline{\alpha}, p^s_{i+2}(\underline{\alpha}), \kappa)$. They will evolve backwards in time so that $\underline{A}$ stays constant, $\underline{\alpha}$ evolves quasi-periodically hence "rigidly", and $p^s_{i+2}(\underline{\alpha})$ evolves to κ while the q–coordinate evolves from κ to $q=p^s_{i+2}(\underline{\alpha})$ (because pq stays constant, see (2.1)). This requires going backward in time by an amount of the order of $T_{\underline{\alpha}} \simeq \overline{g}^{-1}\log\kappa|p^s_{i+2}(\underline{\alpha})|^{-1} \xrightarrow[\underline{\alpha}\to\underline{\alpha}_{i+1}]{} +\infty$.

Therefore there is a sequence $\underline{\alpha}^n \ne \underline{\alpha}_{i+1}$ such that $\underline{\alpha}^n \to \underline{\alpha}_{i+1}$, $p^s_{i+2}(\underline{\alpha}^n) \to 0$ $\underline{A}^s_{i+2}(\underline{\alpha}^n) \to \underline{A}_{i+1}$ *and* $\underline{\alpha}^n - \underline{\omega}T_{\underline{\alpha}^n} \xrightarrow[n\to\infty]{} \underline{\alpha}'_i$, as a consequence of the Diophantine properties of $\underline{\omega}$. So that there is $\tilde{\underline{\alpha}}_{i+1} \stackrel{def}{=} \underline{\alpha}^n$ with n large enough and a point $P_{i+1} = (\underline{A}^s_{i+2}(\tilde{\underline{\alpha}}_{i+1}), \tilde{\underline{\alpha}}_{i+1}, p^s_{i+2}(\tilde{\underline{\alpha}}_{i+1}), \kappa) \in W^s(\underline{A}_{i+2})$ (actually infinitely many) which evolves, backwards in time, from P_{i+1} to a point of B'_i.

Hence we can define $r_{i+1} = |\tilde{\underline{\alpha}}_{i+1} - \underline{\alpha}_{i+1}|$ and ρ_{i+1} small enough so that the backward motion of the points in B_{i+1} enters in due time into B'_i. It follows that the set B_i evolves in time so that all the points of B_{i+1} are on trajectories of points of B_i. Hence all points of $B_{\mathcal{N}}$ will be reached by points starting in B_0.

This completes the proof. All constants can be computed explicitly, even though this is somewhat long and cumbersome. The result is an extremely large diffusion time T (namely $\overline{g}^{-1}$ times the value at $\mathcal{N}$ of a composition of $\mathcal{N}$ exponentials! at least this is the estimate I get after correcting an error in §8 of [CG]: the error was minor but its simple correction leads to substantially worse bounds, see [CG]).

§6. Concluding remarks:

(1) The above proof of Arnold's diffusion is very simple but it leads to a "bad" bound. One can improve the bound by using that there are many invariant tori and one can choose around which to construct a diffusing trajectory (for ε small enough). The bound goes down to $g^{-1}O(2^{\mathcal{N}})$, see [G3].

(2) This is still far from the best bounds in the literature for similar problems, [Be], [Br], obtained by different methods.

(3) The above (obvious) proof is likely to be what Arnold, [A], had in mind when he proposed his example: the example has in common with the above (1.1) *the key feature* of having a coordinate system like the one discussed in §2. The corresponding local Hamilton Jacobi's equation admits a solution "with no exceptions", one says a "gap-less system of coordinates", trivializing the flow: usually when the system is not isochronous or it is not of the Arnold's type there are open regions of phase space where the nice coordinates in which the flow is trivial cannot be defined, see [CG] for what can be done in such cases.

(4) The role of the special coordinates $(\underline{A}_\infty, \underline{\alpha}, p_\infty, q_\infty)$ defined after (2.9) is *essential*: it is clear from the analysis of §5 that one must control the motions near the tori over a very long time span. Since the equations of motion in the original coordinates are non linear this would be impossible in absence of an *exact and simple* solution, as the one given by (3.3): any small perturbation of the initial data would be amplified exponentially in time in an uncontrollable way. The (3.3), instead, confine the expansion to either the p or q coordinate and in a very explicit way. Hence the above is a good example of a non trivial use of the Hamilton–Jacobi's equation.
(5) The method discussed in §5 is a simplified version of a method known as "windowing" described in a early work, [Ea], and developed in more recent works [M], [C]. But it is much less ambitious and detailed and the bounds obtained in the last two papers are far better than the ones we achieve here. I am indebted to P. Lochak and J. Cresson for pointing out the relevance of the latter papers.

Acknowledgments: *This is a contribution to the ISI school on dynamical systems of June 1997 in Torino: I am indebted to G. Gentile and V. Mastropietro for many discussions and suggestions. I thank A. Ambrosetti and E. Vesentini for the invitation to lecture at ISI.*

Appendix A1. Jacobi's map.

This appendix is standard: here it is taken from A9 of [CG] with small changes, to use it for future references.
The theory of Jacobian elliptic functions shows how to perform a complete calculation of the functions, below denoted R, S, in terms of which the canonical Jacobi's coordinates are defined, see [GR] (9.198),(9.153), (9.146), (9.128), (9.197). The result, reported for completeness, is discussed in terms of the pendulum energy:

$$\frac{\dot\varphi^2}{2} + g^2(1-\cos\varphi) = E \tag{A1.1}$$

where the origin in φ is set at the stable equilibrium, to adhere to the notations in the theory of elliptic functions. Setting $u = t(E/2)^{1/2} \equiv \varepsilon^{1/2} g t$, $k^2 = 2g^2/E = \varepsilon^{-1}$ where ε is the *dimensionless* energy so that $\varepsilon = 1$ is the separatrix, let:

$$\mathbf{K}(k) = \int_0^{\pi/2} \frac{d\alpha}{(1-k^2\sin^2\alpha)^{1/2}} \tag{A1.2}$$

We shall use the "standard" notations (*i.e.* those in [GR]) for the Jacobian elliptic integrals *except* for $x(.)$, which is usually denoted $q(.)$, but which we would confuse with the variable q that we want to construct:

$$\begin{aligned} k' =& (1-k^2)^{1/2}, \qquad g_J = g\frac{\pi}{2k\mathbf{K}(k')} = \varepsilon^{1/2} g, \qquad \lambda \equiv \frac{1}{2}\frac{1-k^{1/2}}{1+k^{1/2}} \\ x(k') =& e^{-\pi\mathbf{K}(k)/\mathbf{K}(k')} = \lambda + 2\lambda^5 + 15\lambda^9 + 150\lambda^{13} + 1707\lambda^{17} + \ldots \end{aligned} \tag{A1.3}$$

In terms of the above notations we have, directly from the definitions (*i.e.* from the equations of motion):

$$I(t) = \dot\varphi = -2g\varepsilon^{1/2}\,\mathrm{dn}\,(u,k), \qquad \varphi(t) = 2\,\mathrm{am}\,(tg\varepsilon^{1/2}) \tag{A1.4}$$

which yield, changing the origin for φ to the unstable point to conform with our notations (*i.e.* obtaining $\varphi(t) = 2(\,\mathrm{am}\,(tg\varepsilon^{1/2}) + \pi/2)$), for $I(t) = R(p(t), q(t)), \varphi(t) = S(p(t), q(t))$:

$$R = -2g\varepsilon^{1/2}\frac{\mathrm{dn}\,(iu,k')}{\mathrm{cn}\,(iu,k')}, \quad \sin\frac{S}{2} = \frac{1}{\mathrm{cn}\,(iu,k')}, \quad \cos\frac{S}{2} = i\frac{\mathrm{sn}\,(iu,k')}{\mathrm{cn}\,(iu,k')} \tag{A1.5}$$

Setting $p = e^{-g_J t}, q = x(k')e^{g_J t}$, see [GR], and using $R(p,q) = g_J(-p\partial_p + q\partial_q)S(p,q)$ to evaluate S from R, the quoted basic relations between elliptic integrals imply immediately that the $I(t) = R(p(t),q(t)), \varphi(t) = S(p(t),q(t))$, solve the pendulum equations if:

$$\begin{aligned}
R(p,q) =& -2g_J\Big[\tfrac{p}{1+p^2} + \tfrac{q}{1+q^2} - \textstyle\sum_{n=1}^\infty(-1)^n\tfrac{1+x^{2n-1}}{1-x^{2n-1}}(p^{2n-1}+q^{2n-1})\Big]\\
S(p,q) =& 2\Big[\operatorname{arctg} p - \operatorname{arctg} q - \textstyle\sum_{n=1}^\infty(-1)^n\tfrac{1+x^{2n-1}}{1-x^{2n-1}}\tfrac{(p^{2n-1}-q^{2n-1})}{2n-1}\Big]\\
\sin\tfrac{S(p,q)}{2} =& \tfrac{g_J}{g}\Big[\tfrac{p}{1+p^2} - \tfrac{q}{1+q^2} - \textstyle\sum_{n=1}^\infty(-1)^n\tfrac{1-x^{2n-1}}{1+x^{2n-1}}(p^{2n-1}-q^{2n-1})\Big]\\
\cos\tfrac{S(p,q)}{2} =& -\tfrac{g_J}{2g}\Big[\tfrac{1-p^2}{1+p^2} + \tfrac{1-q^2}{1+q^2} + 2\textstyle\sum_{n=1}^\infty(-1)^n\tfrac{1-x^{2n}}{1+x^{2n}}(p^{2n}+q^{2n})\Big]
\end{aligned}\qquad (A1.6)$$

with $x \equiv pq$. Note that g_J depends on x, and so do k', k: hence the coefficients of the first and of the last two of (A1.6) are also functions of $x = pq$. Furthermore the (dimensionless) energy ε becomes a function $\varepsilon(\xi)$ of $\xi = pq$ defined by inverting the map:

$$\xi = x(k') \equiv x((1-\varepsilon^{-1})^{1/2}) \qquad (A1.7)$$

and the point corresponding to $\varphi = \pi$ and to a dimensionless energy ε, has coordinates:

$$p \equiv 1, \quad q \equiv x(k') \qquad (A1.8)$$

(a rearrangement of (A1.6) showing convergence for $p = 1$ and $|x| < 1$ is exhibited below).

The variables (p,q) defined above are nice and natural: however they are not canonically conjugated to (I,φ): the Jacobian determinant of the map $(p,q) \longleftrightarrow (I,\varphi)$ is not 1. But the Jacobian determinant must be a function $D(x) = \frac{\partial(p,q)}{\partial(I,\varphi)}$ of x alone (*i.e.* of the product pq); then (A1.8) and the equations of motion imply that $D(x)^{-1} = g_J^{-1}\frac{2g^2 d\varepsilon(x)}{dx} = 4g\frac{d\varepsilon^{1/2}}{dx}$.

Therefore one can modify the variables p,q into new variables $(p_J,q_J) = (pF(x), qF(x))$ with F such that the Jacobian $\frac{\partial(p_J,q_J)}{\partial(I,\varphi)} = \frac{\partial(p_J,q_J)}{\partial(p,q)}D(x)$, which is $D(x)\cdot\partial_x(xF^2(x))$, is identically 1. One finds: $F(x) = (4g)^{1/2}(\frac{\varepsilon(x)^{1/2}-1}{x})^{1/2}$.

To invert the map $(p_J,q_J) = (pF(x), qF(x))$ define $x_J \stackrel{def}{=} p_J q_J$ and $G(x_J) \stackrel{def}{=} F(x)^{-1}$ then: $p = p_J G(x_J)$ and $q = q_J G(x_J)$, $x = x_J G^2(x_J)$. The final result is a local canonical map between Jacobi's coordinates (p_J,q_J) and global (I,φ) coordinates:

$$I = R(p_J G(x_J), q_J G(x_J)), \qquad \varphi = S(p_J G(x_J), q_J G(x_J)) \qquad (A1.9)$$

where R,S are defined above, see (A1.6) which are written in a form easily recognized in the elliptic functions tables. The functions R,S can be rewritten in the following form:

$$\begin{aligned}
R(p,q) =& -4g\Big[\textstyle\sum_{m=0}^\infty\big(\tfrac{x^m p}{1+x^{2m}p^2} + \tfrac{x^m q}{1+x^{2m}q^2}\big)\Big]\\
S(p,q) =& 4\big[\textstyle\sum_{m=0}^\infty\big(\operatorname{arctg} x^m p - \operatorname{arctg} x^m q\big)\big]\\
\sin\tfrac{S(p,q)}{2} =& \tfrac{2g_J(x)}{g}\Big[\textstyle\sum_{m=0}^\infty(-1)^m\big(\tfrac{x^m p}{1+x^{2m}p^2} - \tfrac{x^m q}{1+x^{2m}q^2}\big)\Big]\\
\cos\tfrac{S(p,q)}{2} =& \tfrac{g_J(x)}{2g}\Big[1 - 2\textstyle\sum_{m=0}^\infty(-1)^m\big(\tfrac{x^{2m}p^2}{1+x^{2m}p^2} + \tfrac{x^{2m}q^2}{1+x^{2m}q^2}\big)\Big]
\end{aligned}\qquad (A1.10)$$

exhibiting some of the properties of the Jacobi's map in a better way.

One checks that in the (p_J,q_J) variables the pendulum Hamiltonian, (A1.1) has become a function $J(p_J q_J) = 2g^2 + g x_J + O(x_J^2)$. The domain of definition of the map is given by the properties of the elliptic functions or, more restrictively, by the domain of convergence of the above series. It includes a disk of some radius $\rho_J > 0$ around the origin.

The important symmetry $R(p,q) = R(q,p)$ and $S(p,q) = -S(q,p)$ is manifest.

Appendix A2: Dimensional bounds.

We give here two examples of dimensional estimates.

• The bounds (2.5) are obtained by writing (with $\underline{\nu}\in Z^{\ell-1}$):

$$\Phi_0(\underline{\alpha},p_1,q_0)=-\sum_{\substack{\underline{\nu},n,m\\ |\underline{\nu}|+|n-m|>0}} f_{\underline{\nu},n,m}\frac{e^{i\underline{\nu}\cdot\underline{\alpha}}p_1^m q_0^n}{i\underline{\omega}\cdot\underline{\nu}+g(p_1q_0)(n-m)} \tag{A21.1}$$

so that setting $\xi_0''=\xi_0-\frac{\delta_0}{4},\kappa_0''=\kappa_- e^{-\frac{\delta_0}{4}}$ and $\xi_0'=\xi_0-\frac{\delta_0}{2},\kappa_0'=\kappa_0e^{-\frac{\delta_0}{2}}$ (assuming $\ell\ge3$ and using the bound on $|f_{\underline{\nu},m,n}|$ preceding (2.3)):

$$||\Phi_0||_{\xi_0'',\kappa_0''}\le\sum_{\underline{\nu},m,n}\Big(\frac{|\underline{\nu}|^\tau\delta_{m=n}}{C_0}+\frac{\delta_{m\ne n}}{\overline{g}_0}\Big)e^{-\frac14\delta_0|\underline{\nu}|}e^{-\frac14\delta_0(n+m)}\,\varepsilon_0\le B_1\frac{\varepsilon_0}{\Gamma_0\delta_0^{\tau+\ell}} \tag{A2.2}$$

for a suitably chosen B_1; hence (2.5) follows because, for instance, $|\partial_q\Phi_0||_{\xi_0',\kappa_0'}$ can be bounded above by $const\,\kappa_0^{-\frac12}||\Phi_0||_{\xi_0'',\kappa_0''}$ (a typical example of a dimensional estimate).

• By using (2.3) one immediately finds:

$$\begin{aligned}\mathcal{H}_0=&\mathcal{H}_1+\big(\underline{\omega}\cdot\underline{\partial}\,\Phi_0+g_0(p_1q_0)\,\partial\!\!\!/\Phi_0\big)+f_0(\underline{\alpha},p_1,q_0)-\overline{f}_0(p_1\,q_0)+\\&+\big[J_0((p_1+\partial_{q_0}\Phi_0)q_0)-J_0(p_1\,q_0)-J_0'(p_1\,q_0)\,q_0\,\partial_{q_0}\Phi_0\big]+\\&-\big[J_0(p_1(q_0-\partial_{p_1}\Phi_0))-J_0(p_1\,q_0)+J_0'(p_1\,q_0)\,p_1\,\partial_{p_1}\Phi_0\big]+\\&+\big\{f_0(\underline{\alpha},p_1+\partial_{q_0}\Phi_0,q_0)-f_0(\underline{\alpha},p_1,q_0)\big\}-\big\{\overline{f}_0(p_1\,(q_0+\partial_{p_1}\Phi_0))-\overline{f}_0(p_1\,q_0)\big\}\end{aligned} \tag{A2.3}$$

The terms in square and curly brackets can be bounded dimensionally: for instance the first term in square brackets is bounded in the domain $P_{\xi_0-2\delta_0,\kappa_0e^{-2\delta_0}}$ by a bound on the second derivative of J_0 in $P_{\xi_0-\delta_0,\kappa_0e^{-\delta_0}}$ (*i.e.* $2\,\overline{g}_0$ times $\kappa_0^{-1}(1-e^{-\delta_0})^{-1}$ times a bound (on $P_{\xi_0-2\delta_0,\kappa_0e^{-2\delta_0}}$) of the square of $p_1\partial_{p_1}\Phi_0$ or $q_0\partial_{q_0}\Phi_0$ that is given by (2.5)': provided $|p_1\partial_{p_1}\Phi_0|,|q_0\partial_{q_0}\Phi_0|<\delta_0$ in $P_{\xi_0-2\delta_0,\kappa_0e^{-2\delta_0}}$ which gives the condition following (2.7)).

Likewise the terms in curly brackets can be bounded. The final result is (2.7).

References.

[A] Arnold, V.: *Instability of dynamical systems with several degrees of freedom*, Sov. Mathematical Dokl., 5, 581-585, 1966.

[Be] Bessi, U.: it An approach to Arnold's diffusion through the Calculus of Variations, Nonlinear Analysis, 1995.

[Br] Bernard, P.: *Perturbation d'un hamiltonien partiellement hyperbolique*, C.R. Academie des Sciences de Paris, **323**, I, 189–194, 1996.

[C] Cresson, J.: *Symbolic dynamics for homoclinic partially hyperbolic tori and "Arnold diffusion"*, Preprint of Institut de mathematiques de Jussieux, 1997.

[CG] Chierchia, L., Gallavotti, G.: *Drift and diffusion in phase space*, Annales de l' Institut Poincarè, B, **60**, 1–144, 1994.

[Ea] Easton, R.W.: *Orbit structurenear trajectories biasymptotic to invariant tori*, in *Classical mechanics and dynamical systems*, ed. R. Devaney, Z. Nitecki, p. 55–67, Dekker, 1981.

[G1] Gallavotti, G.: The elements of mechanics, Springer, 1983.

[G2] Gallavotti, G.: *Twistless KAM tori*, Communications in Mathematical Physics **164**, 145–156, (1994).

[G3] Gallavotti, G.: *Fast Arnold's diffusion in isochronous systems*, preprint, mp_arc@math.utexas.edu # 97-481, and chao-dyn@xyz.lanl.gov # 9709011.

[M] Marco, J.P.: *Transitions le long des chaines de tores invariants pour les systèmes hamiltoniens analytiques*, Annales de l' Institut Poincaré, **64**, 205–252, 1995.

Internet: Author's preprints downloadable (latest version) at:

`http://chimera.roma1.infn.it`

`http://www.math.rutgers.edu/~giovanni`

Mathematical Physics Preprints (mirror) pages.

e-mail: giovanni@ipparco.roma1.infn.it

Archived also in:
mp_arc@math.utexas.edu # 97-??? and
chao-dyn@xyz.lanl.gov # 970????